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OBJECTIVITY OF THE CONSTITUTIVE EQUATION FOR A MATERIAL WITH MEMORY

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Abstract—The restriction imposed by objectivity on the constitutive functional for a material with memory is well-known. However, the derivation usually given is open to certain objections. A more satisfactory proof is given in this note.

STATEMENT OF THE PROBLEM

We consider the deformation of a material with memory in which a generic particle with initial vector position $\mathbf{X}$ with respect to a fixed origin O moves to vector position $\mathbf{x}(\tau)$ at time τ. Let X_A ($A = 1, 2, 3$) and $x_i(\tau)$ ($i = 1, 2, 3$) be the components of $\mathbf{X}$ and $\mathbf{x}(\tau)$ respectively in a rectangular Cartesian coordinate system x with origin at O. We may regard $\mathbf{X}$ and $\mathbf{x}(\tau)$ as the column matrices formed by X_A and $x_i(\tau)$. We suppose that at and prior to time t_0 the material is in its virgin state.

The deformation gradient matrix, $\mathbf{g}(\tau)$ at time τ, referred to the coordinate system x is defined by

$$\mathbf{g}(\tau) = \|g_{iA}(\tau)\| = \|\partial x_i(\tau)/\partial X_A\|. \tag{1}$$

We make the constitutive assumption that the Cauchy stress matrix $\boldsymbol{\sigma}(t)$ at time t, referred to the coordinate system x, is a symmetric matrix-valued functional of $\mathbf{g}(\tau)$ with compact support $[t_0, t]$:

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\}. \tag{2}$$

Superposition on the assumed deformation of an arbitrary (time-dependent) rotation, or alternatively time-dependent rotation of the reference system x, causes the Cauchy stress matrix $\boldsymbol{\sigma}(t)$ to be rotated by the amount of this rotation at time t; i.e. the constitutive equation is objective. This implies that $\mathbf{F}\{\mathbf{g}(\tau)\}$ must satisfy the relation‡:

$$\mathbf{Q}(t)\mathbf{F}\{\mathbf{g}(\tau)\}\mathbf{Q}^{\dagger}(t) = \mathbf{F}\{\mathbf{Q}(\tau)\mathbf{g}(\tau)\} \tag{3}$$

for all proper orthogonal $\mathbf{Q}(\tau)$ such that $\mathbf{Q}(t_0) = \mathbf{I}$. It is well-known that a necessary and sufficient condition for (3) to be satisfied is that $\mathbf{F}\{\mathbf{g}(\tau)\}$ be expressible in the form

$$\mathbf{F}\{\mathbf{g}(\tau)\} = \mathbf{g}(t)\mathbf{G}\{\mathbf{C}(\tau)\}\mathbf{g}^{\dagger}(t), \tag{4}$$

where $\mathbf{G}$ is a symmetric matrix-valued functional of $\mathbf{C}(\tau)$, the Cauchy strain matrix at time τ, defined by

$$\mathbf{C}(\tau) = \mathbf{g}^{\dagger}(\tau)\mathbf{g}(\tau). \tag{5}$$

Proofs of this result have been given by Green and Rivlin (1957) and by Noll (1958). However neither of them is entirely satisfactory. A much improved version of the proof of

‡Here and throughout this note, a dagger denotes the transpose.

Green and Rivlin was given by Rivlin (1970), but this has the physically unattractive feature that it depends on taking $\mathbf{Q}(\tau) = \mathbf{I}$ except at some particular time $\bar{\tau}$ say, and then allowing $\bar{\tau}$ to range throughout the whole interval $[t_0, t]$. Noll's proof, based on the polar decomposition of $\mathbf{g}(\tau)$, involves a logical fallacy. This is of the same nature as that to which Rivlin and Smith (1987) drew attention in discussing Truesdell's (1960) frequently repeated proof of George Green's (1938) classical result that for an elastic material the strain–energy function depends on the deformation gradient matrix through the Cauchy strain matrix.

In this note a proof is given of the result in (4), which is not open to these objections. It will be apparent that it bears a strong relation to the proof employed by George Green in his discussion of elastic materials.

PROOF OF THE THEOREM‡

We define a functional $\mathbf{F}^*\{\mathbf{g}(\tau)\}$ by

$$\mathbf{F}^*\{\mathbf{g}(\tau)\} = \mathbf{g}^{-1}(t)\mathbf{F}\{\mathbf{g}(\tau)\}[\mathbf{g}^{-1}(t)]^\dagger. \tag{6}$$

Then, from (3), it follows that $\mathbf{F}^*$ must satisfy the restriction

$$\mathbf{F}^*\{\mathbf{g}(\tau)\} = \mathbf{F}^*\{\mathbf{Q}(\tau)\mathbf{g}(\tau)\} \tag{7}$$

for all proper orthogonal $\mathbf{Q}(\tau)$ such that $\mathbf{Q}(t_0) = \mathbf{I}$. This means that $\mathbf{F}^*$ is unchanged if an arbitrary (time-dependent) rotation is superposed on the assumed deformation.

It is well-known that if the Cauchy strain matrix $\mathbf{C}(\tau)$, defined by (5), is specified, then the deformation gradient matrix $\mathbf{g}(\tau)$ is determined uniquely apart from an arbitrary superposed rotation. To see this, suppose that in (5) $\mathbf{C}(\tau)$, with $\mathbf{C}(t_0) = \mathbf{I}$, is specified and let $\bar{\mathbf{g}}(\tau)$ be some choice of $\mathbf{g}(\tau)$ which satisfies (5):

$$\mathbf{C}(\tau) = \bar{\mathbf{g}}^\dagger(\tau)\bar{\mathbf{g}}(\tau). \tag{8}$$

Let

$$\mathbf{g}(\tau) = \mathbf{M}(\tau)\bar{\mathbf{g}}(\tau), \tag{9}$$

where $\mathbf{M}(\tau)$ is a non-singular matrix such that $\mathbf{M}(t_0) = \mathbf{I}$. Then $\mathbf{g}(\tau)$ will satisfy (5) if and only if

$$\mathbf{C}(\tau) = \bar{\mathbf{g}}^\dagger(\tau)\mathbf{M}^\dagger(\tau)\mathbf{M}(\tau)\bar{\mathbf{g}}(\tau). \tag{10}$$

Comparing (8) and (10) we obtain

$$\mathbf{M}^\dagger(\tau)\mathbf{M}(\tau) = \mathbf{I}; \tag{11}$$

$\mathbf{M}(\tau)$ is an orthogonal matrix. Since for any possible deformation $\det \mathbf{g}(\tau) > 0$, it follows that $\det \mathbf{M}(\tau) > 0$; $\mathbf{M}(\tau)$ is a proper orthogonal matrix. Accordingly, for specified $\mathbf{C}(\tau)$, $\mathbf{g}(\tau)$ is uniquely determined apart from a superposed rotation. The condition $\mathbf{M}(t_0) = \mathbf{I}$ then ensures that $\mathbf{g}(t_0) = \mathbf{I}$.

It follows that a necessary and sufficient condition for (7) to be satisfied is that $\mathbf{F}^*\{\mathbf{g}(\tau)\}$ be expressible as a functional of $\mathbf{C}(\tau)$:

$$\mathbf{F}^*\{\mathbf{g}(\tau)\} = \mathbf{G}\{\mathbf{C}(\tau)\}. \tag{12}$$

Equation (6) then yields the result (4).

‡We note that $[\det \mathbf{g}(t)]\mathbf{F}^*\{\mathbf{g}(\tau)\}$ is the second Piola–Kirchhoff stress matrix and (7) is the condition that it be objective.

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